On operators commutative with all invariants for a harmonic oscillator with commensurable frequencies

1995 J. Phys. A: Math. Gen. 28 4407

(http://iopscience.iop.org/0305-4470/28/15/019)

J. Phys. A: Math. Gen. 28 (1995) 4407–4414. Printed in the UK

On operators commutative with all invariants for a harmonic oscillator with commensurable frequencies

A S Nikolaev†

Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

Received 3 February 1995

Abstract. For the **d**-dimensional quantum mechanical harmonic oscillator with **r** commensurability relation between frequencies, **d** independent operators commutative with the oscillator Hamiltonian and all other commutative with Hamiltonian operators are constructed. These operators form a basis in the centre of the algebra of invariants (integrals of motion) for the quantum mechanical oscillator with commensurable frequencies. Not all of these **d** operators have classical analogues. A classical harmonic oscillator only has **d** − **r** commutative with all other invariants. These classical integrals first appeared in the Gustavson work on the Birkhoff normalization. Operators with such properties are of interest for the perturbation theory, since any of them may be (at least formally) continued to become the invariant of the perturbed Hamiltonian.

For a long time the multi-dimensional harmonic oscillator with commensurable frequencies has attracted attention both in classical and quantum mechanics. Both classical and quantum oscillators are well known solvable models in the commensurable as well as in the non-commensurable case. These, and related systems are widely used in various areas of physics. The harmonic oscillator with commensurable frequencies is, in some sense, a purified resonance concept—one of the major topics in modern nonlinear classical mechanics [1]. A well known connection was established between the isotropic harmonic oscillator and the Kepler problem. Anisotropic harmonic oscillators are of great interest in spectroscopy, especially in the investigation of quantum resonance effects (1 : 2 Fermi resonance [6] and its generalizations). Such oscillators were successfully applied to the description of super- and hyperdeformed nuclei ([9] and references therein). Being one of the simplest systems with the hidden symmetry and non-commutative symmetry algebra these systems show a variety of interesting algebraic properties, including quantum algebras [8].

The primary interest in a set of operators, each of them commutative with all commutative-with Hamiltonian operators originates from perturbation theory [2, 4]. The latter, from the algebraic point of view, may be described as a procedure for transforming the Hamiltonian, which differs slightly from the initial exactly solvable one, into the operator belonging to the algebra of invariants for the unperturbed problem. Both the transformation, which preserves equations of motion, and the transformed Hamiltonian will be obtained (at least formally) as a power series in perturbation strength. The transformed Hamiltonian, thus obtained, also has an expansion in powers of generators of this algebra. In such an approach, it is clear, that the operator, commutative with all generators of this algebra,

† E-mail address: nikolaev@mx.ihep.su

will survive under general perturbation, in a sense that it may be extended analytically to become commutative with the perturbed Hamiltonian. As a result, eigenvalues of the perturbed quantum mechanical problem may be classified, in part, by quantum numbers originated from the centre of the algebra of invariants for the unperturbed system. On the other hand, in classical mechanics a search for a new, independent of the Hamiltonian, integral of motion is one of the primary goals of the perturbation theory. Such an integral allows one to reduce the classical system to the lower-dimensional one.

In this paper we describe the construction of operators, commutative with all other operators from the algebra of invariants both for classical and quantum mechanical oscillators with commensurable frequencies. It is shown that for a **d**-dimensional quantum mechanical oscillator with **r** commensurability relation between frequencies there always exists **d** independent operators with such properties. Not all of these operators have a classical analogue. In fact, the corresponding classical oscillator has only $\mathbf{d}-\mathbf{r}$ commutative with all other invariants.

These classical integrals first appeared in the Gustavson paper on the Birkhoff normalization for the Henon–Heiles system [7] as 'integrals of motion for the perturbed system'. We want to note here that the Birkhoff–Gustavson normalization may be naturally extended to the quantum mechanical case, leading to a perturbation scheme equivalent to the standard Rayleigh–Schrödinger perturbation approximation for the polynomially perturbed quantum mechanical harmonic oscillator [10, 11].

We will consider the system described by the Hamiltonian

$$\hat{H}_q(\hat{p},\hat{q}) = \sum_{i=0}^{\mathbf{d}} (\hat{p}_i^2 + \omega_i^2 \hat{q}_i^2) \tag{1}$$

where $\hat{p}_i$ and $\hat{q}_i$ satisfy canonical commutation relations

$$[\hat{p}_i, \hat{q}_j] = -\mathrm{i}\delta_{ij} \tag{2}$$

and other commutators vanish.

Simultaneously with the above quantum mechanical system we will discuss the corresponding classical Hamiltonian

$$H_{cl}(p,q) = \sum_{i=0}^{\mathbf{d}} (p_i^2 + \omega_i^2 q_i^2) \tag{3}$$

with canonically conjugate variables satisfying

$$\{p_i, q_j\} = \delta_{ij}\,. \tag{4}$$

Here Poisson brackets are $\{A, B\} = \sum\limits_k \left(\frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k} - \frac{\partial B}{\partial p_k}\frac{\partial A}{\partial q_k}\right)$.

One may diagonalize both the above Hamiltonians making the canonical transformation to the representation of ladder operators

$$\begin{aligned} \hat{a}_k &= \frac{(\hat{p}_k - \mathrm{i}\omega_k \hat{q}_k)}{\sqrt{2\omega_k}} \qquad k = 1, \ldots, \mathbf{d} \\ \hat{a}_k^+ &= \frac{(\hat{p}_k + \mathrm{i}\omega_k \hat{q}_k)}{\sqrt{2\omega_k}} \end{aligned} \tag{5}$$

and performing an analogous transformation for the classical system (3). These operators (functions) will obey commutation relations

$$[\hat{a}_k, \hat{a}_l^+] = \delta_{kl} \qquad \{a_k, a_l^+\} = \mathrm{i}\delta_{kl} \tag{6}$$

and all other commutators (Poisson brackets) will be identical to zero.

In terms of 'number' operators: $\hat{N}_k = \hat{a}_k^+ \hat{a}_k$ (or, in classical mechanics, actions: $N_k = a_k^+ a_k = \frac{p_k^2 + \omega_k^2 q_k^2}{2\omega_k}$), our Hamiltonians become

$$\hat{H}_q = \sum_{k=0}^{\mathbf{d}} \omega_k \left(\hat{N}_k + \tfrac{1}{2}\right) \qquad H_{cl} = \sum_{k=0}^{\mathbf{d}} \omega_k N_k \tag{7}$$

Now we are in a position to describe generators of symmetry algebras for these systems—a set of independent operators commutative with the initial Hamiltonian. Using terminology borrowed from the classical mechanics we will refer to such an operator as an *integral of motion* or *invariant* of the Hamiltonian.

First of all there are $\mathbf{d}$ commutative with each other invariants, corresponding to individual dimensions

$$\begin{aligned} &\hat{N}_i = \hat{a}_i^+ \hat{a}_i \qquad i, j = 1, \ldots, \mathbf{d} \\ &[\hat{N}_i, \hat{N}_j] = 0 \end{aligned} \tag{8}$$

where $\hat{H}_q$ is a function of them. The same invariants in the classical case are known as *actions*.

We will now distinguish between the degenerate (the resonance in the classical mechanics) and non-degenerate case. One has **r**th-fold degeneracy (the resonance condition) whenever $\mathbf{d}$ frequencies ω_i, are connected by $\mathbf{r}$ (and only $\mathbf{r}$) linearly independent relations of commensurability

$$\sum_{k=1}^{\mathbf{d}} D_{ik} \omega_k = 0 \qquad \begin{aligned} i &= 1, \ldots, \mathbf{r} \\ k &= 1, \ldots, \mathbf{d} \end{aligned} \tag{9}$$

where all D_{ik} are integers†. It will be convenient for us to interpret the above relations as an existence of $\mathbf{r}$ integer-component vectors $\vec{D}_i$ in $\mathbf{d}$-dimensional linear space, such that $(\vec{D}_i, \vec{\omega}) = 0$. If all ω_i are rationally independent, the problem is non-degenerate (non-resonance).

In this non-degenerate case operators $\hat{N}_k$ (functions N_k) form the complete system of $\mathbf{d}$ invariants for the anisotropic harmonic oscillator with incommensurable frequencies. In this case the symmetry algebra is commutative, and all eigenvalues of $\hat{H}$ are non-degenerate and classified by $\hat{N}_k$.

But the degenerate harmonic oscillator has some hidden symmetry, which reveals in the existence of additional non-commutative invariants $\hat{K}_i$ and $\hat{K}_i^+$ [3], where

$$\hat{K}_i = \prod_{k=1}^{\mathbf{d}} \hat{b}_k^{|D_{ik}|}(\operatorname{sign} D_{ik}) \qquad \hat{b}_k(l) = \begin{cases} \hat{a}_k^+ & l = +1 \\ \hat{a}_k & l = -1 \end{cases} \qquad i = 1, \ldots, \mathbf{r}. \tag{10}$$

† Otherwise ergodic on the energy surface, a phase-space trajectory of the classical oscillator with commensurability relations between frequencies will be bounded to lower-dimensional tori. In the coordinate space of the two-dimensional oscillator, for example, this leads to the famous Lissajous figures. The quantum mechanical oscillator with commensurable frequencies will have degenerate energy levels.

One can easily verify the following properties of these non-Hermitian operators:

$$
\begin{aligned}
&[\hat{K}_i, \hat{H}_0] = 0 \qquad i, j = 1, \ldots, \mathbf{r}, \\
&[\hat{K}_i^+, \hat{H}_0] = 0 \\
&[\hat{K}_i, \hat{K}_j] = \hat{P}_{ij} \\
&[\hat{K}_i, \hat{K}_j^+] = \hat{Q}_{ij} \\
&[\hat{K}_i, \hat{N}_k] = -D_{ik}\hat{K}_i \qquad k = 1, \ldots, \mathbf{d}, \\
&[\hat{K}_i^+, \hat{N}_k] = D_{ik}\hat{K}_i^+
\end{aligned} \tag{11}
$$

where $\hat{P}_{ij}$, $\hat{Q}_{ij}$, are some polynomials of the above invariants. Obvious changes must be made for the classical oscillator.

In general, not all of these invariants ($\hat{N}$'s, $\hat{K}$'s and $\hat{K}^+$'s) are independent. Here we will neither construct the maximal set of independent Hermitian invariants, nor explicitly establish their algebraic properties. We only note that in classical mechanics it is known (at least for the two-dimensional case) that, using a nonlinear canonical transformation, an isomorphism may be established between the above algebra and $u(2)$-symmetry algebra of the isotropic two-dimensional harmonic oscillator [5]. A quantum mechanical situation is more complicated, and nonlinear extensions or quantum deformations of this Lie algebra arise [8].

Instead, here we will focus on the construction of operators belonging to the centre of the above algebra. These operators are commutative with all possible invariants of resonance harmonic oscillator.

We are seeking for analytical (operator) function $\hat{I}(\hat{a}_1^+, \hat{a}_1, \ldots, \hat{a}_\mathbf{d}^+, \hat{a}_\mathbf{d})$, commutative with all the above generators

$$
\begin{aligned}
&[\hat{N}_k, \hat{I}] = 0 \qquad k = 1, \ldots, \mathbf{d} \\
&[\hat{K}_i, \hat{I}] = 0 \qquad i = 1, \ldots, \mathbf{r}\,.
\end{aligned} \tag{12}
$$

We suppose that the (operator) function $\hat{I}$ has an expansion in increasing powers of operators $\hat{a}_k^+$ and $\hat{a}_k$

$$
\hat{I} = \sum I_{\substack{m_1,\ldots,m_d \\ n_1,\ldots,n_d}} \hat{a}_1^{+m_1} \ldots \hat{a}_\mathbf{d}^{+m_d} \hat{a}_1^{n_1} \ldots \hat{a}_\mathbf{d}^{n_d} \tag{13}
$$

where all monomials in this expansion are reduced to the Wick normal ordering. If one takes into account the identity

$$
[\hat{N}_k, \hat{a}_k^{+m}\hat{a}_k^n] = (m-n)\hat{a}_k^{+m}\hat{a}_k^n \tag{14}
$$

it is clear that in order to satisfy the first part of conditions (12), $\hat{I}$ *must* be a function of $\hat{N}_k$ only

$$
\hat{I} = \hat{I}(\hat{N}_1, \ldots, \hat{N}_\mathbf{d})\,.
$$

This is a general condition for the diagonal operator.

For classical oscillator (3) we have analogous arguments, leading to the same conclusion. Identity (14) takes the following form:

$$
\{N_k, a_k^{+m}a_k^n\} = \mathrm{i}(m-n)a_k^{+m}a_k^n\,. \tag{15}
$$

We now proceed to the discussion of the second part of conditions (12)

$$
\left[\hat{I}(\hat{N}_1, \ldots, \hat{N}_\mathbf{d}), \prod_{k=1}^{\mathbf{d}} \hat{b}_k^{|D_{ik}|}(\operatorname{sign} D_{ik})\right] = 0\,. \tag{16}
$$

We will need the following general commutation relations:

$$\hat{F}(\hat{N})\hat{a}^{+m} = \hat{a}^{+m}\hat{F}(\hat{N}+m)$$
$$\hat{F}(\hat{N})\hat{a}^{m} = \hat{a}^{m}\hat{F}(\hat{N}-m) \tag{17}$$

which may be easily proved for any function F if one compares matrix elements of these operators.

Using the above relations, condition (16) will be reduced to

$$[\hat{I}(\hat{N}_1,\ldots,\hat{N}_{\mathbf{d}}),\hat{K}_i] = \hat{K}_i\big(\hat{I}(\hat{N}_1+D_{i1},\ldots,\hat{N}_{\mathbf{d}}+D_{i\mathbf{d}}) - \hat{I}(\hat{N}_1,\ldots,\hat{N}_{\mathbf{d}})\big) = 0\,. \tag{18}$$

Corresponding classical Poisson brackets will look like

$$\{I(N_1,\ldots,N_{\mathbf{d}}),K_i\} = \mathrm{i}\left(\sum_{k=0}^{\mathbf{d}} D_{ik}\frac{\partial I}{\partial N_k}\right)K_i = 0\,. \tag{19}$$

Therefore in order to find the general form of the operator, commutative with all $\hat{N}_k$ and $\hat{K}_i$, we must solve the following system of $\mathbf{r}$ functional equations:

$$\hat{I}(\vec{N}+\vec{D}_i) = \hat{I}(\vec{N}) \qquad i = 1,\ldots,\mathbf{r} \tag{20}$$

where we use an obvious vector notation. This may be done in the following way.

As has already been mentioned, we interpret $\mathbf{r}$ commensurability relations (9) as the existence of $\mathbf{r}$ integer-component independent vectors $\vec{D}_i$ in $\mathbf{d}$-dimensional linear space. We can always perform linear transformation of variables:

$$\tilde{N} = C\vec{N} \qquad \tilde{D}_i = C\vec{D}_i \qquad i = 1,\ldots,\mathbf{d} \tag{21}$$

where C is any $\mathbf{d}\times\mathbf{d}$ invertible matrix. This transformation preserves the structure of the above functional system

$$\hat{I}(C^{-1}(\tilde{N}+\tilde{D}_i)) = \hat{I}(C^{-1}\tilde{N})\,. \tag{22}$$

We will choose matrix C, so that $\tilde{D}_i$ will become the first $\mathbf{r}$ new basis vectors. Functional equations after such transformation will take the following simple form:

$$\begin{aligned}&\hat{I}(\tilde{N}_1+1,\ldots,\tilde{N}_{\mathbf{r}},\ldots,\tilde{N}_{\mathbf{d}}) = \hat{I}(\tilde{N}_1,\ldots,\tilde{N}_{\mathbf{d}})\\ &\vdots\\ &\hat{I}(\tilde{N}_1,\ldots\tilde{N}_{\mathbf{r}}+1,\ldots,\tilde{N}_{\mathbf{d}}) = \hat{I}(\tilde{N}_1,\ldots,\tilde{N}_{\mathbf{d}})\,.\end{aligned} \tag{23}$$

It is clear from the above that (operator) functions $\hat{I}$ are independently periodic in the first $\mathbf{r}$ arguments, and therefore may be expanded in Fourier series

$$\hat{I}(C^{-1}\tilde{N}) = \sum_{k_1,\ldots,k_r=-\infty}^{+\infty} f_{k_1,\ldots,k_r}(\tilde{N}_{\mathbf{r}+1},\ldots,\tilde{N}_{\mathbf{d}})e^{2\pi\mathrm{i}\sum_{i=1}^{\mathbf{r}}k_i\tilde{N}_i} \tag{24}$$

where $f_{k_1,\ldots,k_r}(\tilde{N}_{\mathbf{r}+1},\ldots,\tilde{N}_{\mathbf{d}})$ are arbitrary analytical functions of $\mathbf{d}-\mathbf{r}$ (operator) variables $\tilde{N}$.

Performing the inverse transformation, we find the general solution of (20) in the form

$$\hat{I}(\vec{N}) = \sum_{k_1,\ldots,k_r=-\infty}^{+\infty} f_{k_1,\ldots,k_r}(\hat{\Gamma}_{\mathbf{r}+1},\ldots,\hat{\Gamma}_{\mathbf{d}})\exp\left(2\pi\mathrm{i}\sum_{i=1}^{\mathbf{r}}k_i\hat{\Gamma}_i\right) \tag{25}$$

where we denote

$$\hat{\Gamma}_i = \sum_{m=0}^{\mathbf{d}}\alpha_{im}\hat{N}_m \qquad i = 1,\ldots,\mathbf{d} \tag{26}$$

and α_{im} is a set of vectors, satisfying the following system of linear equations (note the different ranges of indices):

$$\sum_{m=0}^{\mathbf{d}} \alpha_{im} D_{km} = \delta_{ik} \qquad \begin{array}{l} i = 1, \ldots, \mathbf{d} \\ k = 1, \ldots, \mathbf{r}. \end{array} \tag{27}$$

Using the vector notation we can easily see that the above system is consistent and has solutions. Indeed, one may always add $\mathbf{d}-\mathbf{r}$ independent vectors to our $\mathbf{r}$ commensurability vectors $\vec{D}_i$ in order to costruct the complete, in general not orthogonal basis. From linear system (27) it is clear that $\vec{\alpha}_i$ must be chosen equal to vectors of the basis dual to the above constructed basis.

For the classical oscillator the same transformation, applied to a system of partial derivative differential equations, reduces this system to

$$\frac{\partial I}{\partial \tilde{N}_k} = 0 \qquad k = 1, \ldots, \mathbf{r} \tag{28}$$

with the obvious solution

$$I(N_1, \ldots, N_{\mathbf{d}}) = I(\Gamma_{\mathbf{r}+1}, \ldots, \Gamma_{\mathbf{d}}) \tag{29}$$

only. Here we use the same notation as for quantum mechanical case.

From equation (25) it is clear that for the quantum mechanical harmonic oscillator there always exist $\mathbf{d}$ independent invariants, commutative with all other invariants. These invariants may be subdivided into two distinct groups:

- $\mathbf{d}-\mathbf{r}$ invariants $\hat{\Gamma}_i = \sum_{m=0}^{\mathbf{d}} \alpha_{im} \hat{N}_m$, $i = \mathbf{r}+1, \ldots, \mathbf{d}$. These invariants were implicitly found by Gustavson [7] for the classical oscillator. Sometimes such operators are called the first-order Casimirs.
- New $\mathbf{r}$ periodic invariants $\exp(2\pi \mathrm{i} \hat{\Gamma}_k)$, $k = 1, \ldots, \mathbf{r}$ specific to the quantum mechanical harmonic oscillator and have no classical analogues.

These $\mathbf{d}$ operators form the basis in the centre of algebra of invariants for the harmonic oscillator.

One can easily see that matrix forms of the above constructed operators are diagonal. Moreover, using the fact that quantum numbers $\vec{N}$ for two eigenvectors of $\hat{H}$ with the same eigenvalue may differ only in the integer coefficient combination of $\vec{D}_i$, we may conclude that any degenerate eigensubspace of oscillator Hamiltonian is degenerate eigensubspace of all $\mathbf{d}$ invariants also. This is not surprising since all these invariants will survive under *general* perturbation [11].

Example 1. Two-dimensional isotropic oscillator

The symmetry algebra for the quantum mechanical isotropic oscillator (commensurability relation: $\omega_1 - \omega_2 = 0$) with the Hamiltonian

$$\hat{H} = \hat{p}_1^2 + \hat{q}_1^2 + \hat{p}_2^2 + \hat{q}_2^2 = \hat{N}_1 + \hat{N}_2 + 1$$

is generated by operators

$$\hat{N}_1 = \hat{a}_1^+ \hat{a}_1 \qquad \hat{N}_2 = \hat{a}_2^+ \hat{a}_2$$
$$\hat{K}_1 = \hat{a}_1^+ \hat{a}_2 \qquad \hat{K}_1^+ = \hat{a}_2^+ \hat{a}_1 .$$

According to the above described construction, the basis of the centre of algebra of invariants is formed by two operators. The first one is the Gustavson invariant

$$\hat{\Gamma}_1 = \hat{N}_1 + \hat{N}_2 .$$

And the second one is an additional Hermitian quantum mechanical invariant of the form

$$\cos\pi(\hat{N}_1 - \hat{N}_2)\,.$$

One can easily see that matrix elements of this operator coincide with those of the parity operator corresponding to the central symmetry transformation

$$q_1 \to -q_1 \qquad q_2 \to -q_2\,.$$

The same conclusion is valid for the **d** dimensional isotropic harmonic oscillator. For such system $\mathbf{d}-1$ additional quantum invariants coincide with parities for $\mathbf{d}-1$ central symmetry transformations in $\mathbf{d}-1$ orthogonal planes in the **d**-dimensional coordinate space.

Example 2. Two-dimensional 1:2 resonance—Fermi oscillator

The 1:2 Fermi oscillator is described by the Hamiltonian

$$\hat{H} = \hat{p}_1^2 + \hat{q}_1^2 + \hat{p}_2^2 + 4\hat{q}_2^2 = \hat{N}_1 + 2\hat{N}_2 + \tfrac{3}{2}\,.$$

Now the commensurability relation between frequencies will be

$$2\omega_1 - \omega_2 = 0\,.$$

The symmetry algebra is generated by

$$\hat{N}_1 = \hat{a}_1^+\hat{a}_1 \qquad \hat{N}_2 = \hat{a}_2^+\hat{a}_2$$
$$\hat{K}_1 = \hat{a}_1^{+2}\hat{a}_2 \qquad \hat{K}_1^+ = \hat{a}_2^+\hat{a}_1^2\,.$$

As in the previous example the Gustavson invariant coincides with the Hamiltonian

$$\hat{\Gamma}_1 = \hat{N}_1 + 2\hat{N}_2\,.$$

And the additional quantum mechanical invariant will be

$$\exp\tfrac{2}{5}\pi\mathrm{i}(2\hat{N}_1 - \hat{N}_2)\,.$$

Hermitian and anti-Hermitian parts of this invariant corresponds to two (not independent) observables. Matrix elements of these observables will take five different values. One may treat these observables as some 'generalized parities' for the Fermi oscillator.

In summary, for a **d**-dimensional quantum mechanical oscillator with **r** commensurability relations between frequencies we construct **d** independent operators commutative with the oscillator Hamiltonian and all other commutative with Hamiltonian operators. These operators form the basis for the centre of the algebra of invariants for the quantum oscillator with commensurable frequencies. Among them only **d**−**r** invariants have the classical analogue. These classical invariants were implicitly found by Gustavson [7]. Other **r** specific to quantum mechanics invariants have the meaning of 'generalized parities'. Operators with such properties are of interest for the perturbation theory, since any of them may be (at least formally) continued to remain an integral of motion under general perturbation of the oscillator Hamiltonian.

Acknowledgments

Author gratefully acknowledges I V Mandrichenko for fruitful discussions. This work was supported in part by the Russian Foundation for Fundamental Investigations under contract RFFI-93-02-3733.

References

[1] Lightenberg A and Libermann M 1983 *Regular and Stochastic Motions* (Berlin: Springer)

[2] Cary J 1981 *Phys. Rep.* **79** 129

[3] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basel: Birkhäuser)

[4] Klein D J 1974 *J. Chem. Phys.* **61** 786

[5] Jauch J M and Hill E L 1940 *Phys. Rev.* **57** 641

[6] Fermi E 1931 *Z. Phys.* **71** 250

[7] Gustavson F G 1966 *Astron. J.* **71** 670

[8] Bonatsos D *et al* 1995 Symmetry algebra of the planar anisotropic quantum harmonic oscillator with rational ratio of frequencies *Preprint* THES-TP/93-13, available as hep-th-9402099

[9] Bonatsos D *et al* 1995 The symmetry algebra of the N-dimensional anisotropic quantum harmonic oscillator with rational frequencies and the Nilsson model *Preprint* THES-TP/26-94, available as hep-th-9411218

[10] Eckhardt B 1986 *J. Phys. A: Math. Gen.* **19** 2961

[11] Nikolaev A S 1995 On the diagonalization of quantum Birkhoff–Gustavson normal form *Preprint* IHEP 94-143